

# From symplectic groupoids to double structures

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## Foreword

These notes are an introduction to symplectic groupoids and the double structures associated with them. The treatment is intended to lie about midway between the original account of Coste, Dazord and Weinstein [4], which relied on effective use of the symplectic structures, and the account in my book [18], which showed, on the level of Poisson groupoids, that the basic results of the theory follow from ‘categorical’ compatibility conditions between the associated Lie algebroid and Lie groupoid structures. (See §5 for more details.)

The reader needs to know only the most basic ideas of symplectic geometry, Lie groups, and vector bundles. Conventions are recalled in §1.

In particular, no familiarity with double structures is assumed. Instead the cotangent groupoid — perhaps one of the hardest ideas to assimilate for someone new to this theory — is introduced as a consequence of the isomorphism between the tangent and cotangent bundles. That then leads to the general concept of a  $\mathcal{VB}$ -groupoid and its duality.

These notes are based on lectures given at the School on *Geometric, Algebraic and Topological Methods for Quantum Field Theory* in Villa de Leyva, Colombia, in July 2013. I am very glad to have had the opportunity to give these lectures, and I thank the organizers most heartily for involving me in a School of such vitality and openness. I want to particularly thank Alexander Cardona for his generous hospitality and for looking after me so well.

These notes aim to introduce the reader to certain important, and relatively new, ideas quickly; accordingly they omit much standard material. All the main results here are known, but the approach has some new features. Some references which provide alternative treatments and more detail are given at the end.

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## 1 Background: Poisson structures, Lie algebroids, Lie groupoids

**Definition 1.1.** A *Poisson structure* on a manifold  $P$  is a bracket of smooth functions

$$\{ , \}: C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P)$$

with respect to which  $C^\infty(P)$  is an  $\mathbb{R}$ -Lie algebra, and such that for all  $f_1, f_2, f_3 \in C^\infty(P)$ ,

$$\{f_1, f_2 f_3\} = f_2 \{f_1, f_3\} + \{f_1, f_2\} f_3. \quad (1)$$

The bracket  $\{f_1, f_2\}$  depends only on  $df_1$  and  $df_2$  and we define a 2-vector field  $\pi$  on  $P$  by  $\pi(w_1 df_1, w_2 df_2) = w_1 w_2 \{f_1, f_2\}$ . This is the *Poisson tensor* and in practice we use  $\pi$  to denote a Poisson structure.

For a symplectic manifold  $(M, \omega)$ , there is an *associated Poisson structure* on  $M$  defined by  $\{f_1, f_2\} = \omega((df_1)^\sharp, (df_2)^\sharp)$ , where  $\varphi \mapsto \varphi^\sharp$ ,  $T^*M \rightarrow TM$  is the inverse of the map  $\omega^\flat: TM \rightarrow T^*M$ ,  $\langle \omega^\flat(X), Y \rangle = -\omega(X, Y)$ .

Returning to a Poisson manifold  $P$ , we define  $\pi^\#: T^*P \rightarrow TP$ , the *Poisson anchor*, by

$$\langle \varphi, \pi^\#(\psi) \rangle = \pi(\psi, \varphi). \quad (2)$$

If  $\pi^\#$  is an isomorphism of vector bundles, then  $\omega(X, Y) = \langle (\pi^\#)^{-1}(Y), X \rangle$  is a symplectic structure on  $P$ , and the associated Poisson structure is the given one.

Define a bracket of 1-forms by

$$[\varphi, \psi] = \mathfrak{L}_{\varphi^\#}(\psi) - \mathfrak{L}_{\psi^\#}(\varphi) - d(\pi(\varphi, \psi)), \quad (3)$$

where again  $\varphi^\# = \pi^\#(\varphi)$ . The bracket of 1-forms makes  $T^*P$  a Lie algebroid with anchor  $\pi^\#$ .

**Definition 1.2.** A *Lie algebroid* is a vector bundle  $A$  on base  $M$  together with a bracket  $[\cdot, \cdot]: \Gamma A \times \Gamma A \rightarrow \Gamma A$ , with respect to which  $\Gamma A$  is an  $\mathbb{R}$ -Lie algebra, and a vector bundle map  $a: A \rightarrow TM$ , called the *anchor of  $A$* , such that

$$[X, fY] = f[X, Y] + a(X)(f)Y, \quad a[X, Y] = [aX, aY], \quad (4)$$

for  $X, Y \in \Gamma A$  and  $f \in C^\infty(M)$ .

If  $A'$  is also a Lie algebroid on  $M$  then a *base-preserving morphism of Lie algebroids* is a morphism of vector bundles  $\varphi: A \rightarrow A'$  such that  $a' \circ \varphi = a$  and  $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$  for all  $X, Y \in \Gamma A$ .

For any vector bundle  $A$  on  $M$ , sections  $X \in \Gamma A$  induce (fibrewise) linear maps  $\ell_X: A^* \rightarrow \mathbb{R}$  by  $\ell_X(\varphi) = \langle \varphi, X(q_*(\varphi)) \rangle$  where  $q_*$  is the bundle projection for  $A^*$ . Every fibrewise linear map  $A^* \rightarrow \mathbb{R}$  is  $\ell_X$  for a unique  $X \in \Gamma A$ .

When  $A$  is a Lie algebroid, there is a unique Poisson structure on the manifold  $A^*$ , called the *Lie-Poisson structure corresponding to  $A$* , such that

$$\{\ell_X, \ell_Y\} = \ell_{[X, Y]}, \quad \{\ell_X, f \circ q_*\} = a(X)(f) \circ q_*, \quad \{f_1 \circ q_*, f_2 \circ q_*\} = 0, \quad (5)$$

for  $X, Y \in \Gamma A$ ,  $f \in C^\infty(M)$ . It is not clear that this defines the bracket for all pairs of smooth functions. To bypass this, consider the Poisson tensor for  $A^*$ , and note that every 1-form is the sum of  $d\ell_X$  for some  $X \in \Gamma A$  and  $q_*^*df$  for some  $f \in C^\infty(M)$ . Conversely, if  $E$  is a vector bundle on  $M$  with a Poisson structure for which: (i) the bracket of fibrewise linear functions is fibrewise linear, (ii) the bracket of a fibrewise linear function with a pullback function is a pullback function, and (iii) the bracket of two pullback functions is zero, then there is a unique Lie algebroid structure on  $E^*$  for which this is the Lie-Poisson structure.

For a Lie algebra  $\mathfrak{g}$ , the second and third definitions in (5) are vacuous. However the first can be extended to any pair of smooth functions. Let  $f: \mathfrak{g}^* \rightarrow \mathbb{R}$  be a smooth function on  $\mathfrak{g}^*$ . Using an elementary notion of derivative,  $D(f)(\varphi)$ , for each  $\varphi \in \mathfrak{g}^*$ , is a linear map  $\mathfrak{g}^* \rightarrow \mathbb{R}$  (the ‘directional derivative’) and can thus be identified with an element of  $\mathfrak{g}$ . Since a Poisson bracket depends only on the derivatives of its arguments, we have

$$\{f_1, f_2\}(\varphi) = \langle \varphi, [D(f_1)(\varphi), D(f_2)(\varphi)] \rangle. \quad (6)$$

A map  $\mu: P \rightarrow Q$  of Poisson manifolds is a *Poisson map* if  $\{f_1 \circ \mu, f_2 \circ \mu\}_P = \{f_1, f_2\}_Q \circ \mu$  for all  $f_1, f_2 \in C^\infty(Q)$ . We will need Poisson maps in §4.

## Lie groupoids

**Definition 1.3.** A *Lie groupoid* consists of a manifold  $M$  of *points*, or *objects*, and a manifold  $\mathcal{G}$  of *arrows*, together with surjective submersions  $\alpha, \beta: \mathcal{G} \rightarrow M$ , called respectively the *source* and *target* maps, and an injective immersion  $1: M \rightarrow \mathcal{G}$ , written  $m \mapsto 1_m$ , together with a smooth *multiplication* or *composition*,  $(h, g) \mapsto hg$ ,  $\mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G}$ , defined on

$$\mathcal{G} \times_M \mathcal{G} = \{(h, g) \in \mathcal{G} \times \mathcal{G} \mid \alpha(h) = \beta(g)\}.$$

These maps are subject to modified forms of the group axioms: for all  $h, g, f \in \mathcal{G}$  such that  $\alpha(h) = \beta(g)$  and  $\alpha(g) = \beta(f)$ , we have  $h(gf) = (hg)f$ ; for each  $g \in \mathcal{G}$  we have  $1_n g = g$  and  $g 1_m = g$  where  $n = \beta(g)$ ,  $m = \alpha(g)$ ; finally, for each  $g \in \mathcal{G}$  there is a  $g^{-1} \in \mathcal{G}$  such that  $g^{-1}g = 1_m$  and  $gg^{-1} = 1_n$ . We often write  $\mathcal{G} \rightrightarrows M$  to indicate a Lie groupoid.

A *morphism of Lie groupoids* consists of two smooth maps  $F: \mathcal{G} \rightarrow \mathcal{G}'$  and  $f: M \rightarrow M'$  which commute with the sources and targets, for which  $F(1_m) = 1'_{f(m)}$  for  $m \in M$ , and for which  $F(hg) = F(h)F(g)$  when  $\alpha(h) = \beta(g)$ . If  $M' = M$  and  $f$  is the identity map, we say that  $F$  is *over*  $M$ .

In these lectures we are concerned with symplectic groupoids, rather than the general theory. We include here only four important examples.

**Examples 1.4. (i)** For any manifold  $M$  the Cartesian square  $M \times M$  is a Lie groupoid with  $\alpha(n, m) = m$ ,  $\beta(n, m) = n$ ,  $1_m = (m, m)$  and multiplication  $(p, n)(n, m) = (p, m)$ . This is the *pair groupoid* on  $M$ .

**(ii)** For a connected manifold  $M$  the *fundamental groupoid*  $\Pi(M)$  is a Lie groupoid on  $M$ . The arrows are homotopy classes, with endpoints fixed, of piecewise smooth paths in  $M$ , the source and target maps give the start-point and end-point, and the groupoid multiplication is concatenation. The identity elements are the classes of constant paths. The map  $\chi: \Pi(M) \rightarrow M \times M$  which sends the class of a path  $p(t)$  to  $(p(1), p(0))$  is a morphism of Lie groupoids over  $M$ .

**(iii)** Let  $E$  be a vector bundle on  $M$ . Denote by  $\Phi(E)$  the set of all linear isomorphisms  $\xi: E_m \rightarrow E_n$  between fibres of  $E$ . The local triviality of  $E$  can be used to give  $\Phi(E)$  a manifold structure. Define source and target  $\alpha(\xi) = m$  and  $\beta(\xi) = n$ , for  $\xi$  as above, and write  $1_m$  for the identity map  $E_m \rightarrow E_m$ . Then the standard composition of maps makes  $\Phi(E)$  a Lie groupoid on base  $M$ , the *frame groupoid* of  $E$ .

**(iv)** Let  $G$  be a Lie group and let  $G \times M \rightarrow M$  be a smooth action of  $G$  on a manifold  $M$ . Write  $\mathcal{G} = G \times M$ , the product manifold. Define  $\alpha: \mathcal{G} \rightarrow M$  by  $\alpha(g, m) = m$  and  $\beta: \mathcal{G} \rightarrow M$  by  $\beta(g, m) = gm$ . For  $m \in M$  write  $1_m = (1, m)$  and define

$$(h, n)(g, m) = (hg, m)$$

when  $n = gm$ . Then  $\mathcal{G}$  is a Lie groupoid on base  $M$ , the *action groupoid* corresponding to the given action. We write  $G \triangleleft M$  for  $\mathcal{G}$ ; many authors write  $G \ltimes M$ .

We now describe the Lie algebroid of a Lie groupoid. The construction follows the same general procedure as the construction of the Lie algebra of a Lie group, though we define the bracket using right-invariant vector fields rather than the more usual left-invariant vector fields.

For a Lie group  $G$  the Lie algebra  $\mathfrak{g}$  may be defined as  $T_1(G)$ , the tangent space at the identity element, with the bracket on  $\mathfrak{g}$  defined by identifying elements of  $\mathfrak{g}$  with right-invariant vector fields.

For a Lie groupoid  $\mathcal{G} \rightrightarrows M$  there is a regular submanifold of identity elements  $\{1_m \mid m \in M\} \subseteq \mathcal{G}$ , which we identify with  $M$ . Write  $T_M\mathcal{G}$  for the restriction of the tangent bundle  $T\mathcal{G}$  to  $M$ . The fibres of  $T_M\mathcal{G}$  are  $T_{1_m}\mathcal{G}$  for  $m \in M$ .

Because the multiplication in  $\mathcal{G}$  is only defined on  $\mathcal{G} \times_M \mathcal{G}$ , right-translations are only defined on the fibres of the source map. For  $g \in \mathcal{G}$  we have

$$R_g: \alpha^{-1}(\beta g) \rightarrow \alpha^{-1}(\alpha g), \quad h \mapsto hg.$$

In order to define right-invariance for vector fields on  $\mathcal{G}$ , we therefore consider only vector fields which are vertical with respect to the source map. Thus:

**Definition 1.5.** A vector field  $\xi$  on  $\mathcal{G}$  is *right-invariant* if  $T(\alpha)(\xi(g)) = 0$  for all  $g \in \mathcal{G}$ , and  $T(R_g)(\xi(h)) = \xi(hg)$  for all  $(h, g) \in \mathcal{G} \times_M \mathcal{G}$ .

Note that for a smooth map  $f: M \rightarrow N$  we usually write  $T(f)$  for the tangent map  $TM \rightarrow TN$ , reserving  $df$  for when  $f$  is real-valued.

The second condition is equivalent to  $\xi(g) = T(R_g)(\xi(1_{\beta g}))$  for all  $g \in \mathcal{G}$ . If the analogy with the Lie algebra of a Lie group is to hold good, it should be sufficient to consider the restrictions of right-invariant vector fields to the manifold of identity elements. We therefore define

$$A\mathcal{G} = \ker(T(\alpha): T\mathcal{G} \rightarrow TM) \cap T_M\mathcal{G}.$$

That is,  $A\mathcal{G}$  is a vector bundle on  $M$ , and the fibre over  $m \in M$  is  $T_{1_m}(\alpha^{-1}(m))$ .

If  $X$  is a section of  $A\mathcal{G}$  we can define a right-invariant vector field  $\overrightarrow{X}$  on  $\mathcal{G}$  by  $\overrightarrow{X}(g) = T(R_g)(X(\beta g))$ . Conversely, every right-invariant vector field is  $\overrightarrow{X}$  for some  $X \in \Gamma A\mathcal{G}$ .

We next need to show that the bracket of right-invariant vector fields is a right-invariant vector field. This follows, as in the case of Lie groups, from the fact that the bracket of projectable vector fields is projectable: note that a vector field  $\xi$  on  $\mathcal{G}$  is right-invariant if and only if it projects to zero under  $\alpha$  and for each  $g \in \mathcal{G}$ , the restriction of  $\xi$  to  $\alpha^{-1}(\beta g)$  projects under  $R_g$  to the restriction of  $\xi$  to  $\alpha^{-1}(\alpha g)$ . We therefore have a bracket on  $\Gamma A\mathcal{G}$  such that, for all  $X, Y \in \Gamma A\mathcal{G}$ ,

$$\overrightarrow{[X, Y]} = [\overrightarrow{X}, \overrightarrow{Y}].$$

Lastly, consider  $[X, fY]$  where  $f \in C^\infty(M)$ . Clearly  $\overrightarrow{fY} = (f \circ \beta)\overrightarrow{Y}$ . Using the Leibniz identity for vector fields on  $\mathcal{G}$ , we therefore get that

$$[X, fY] = f[X, Y] + a(X)(f)Y,$$

where  $a: A\mathcal{G} \rightarrow TM$  is the restriction of  $T(\beta): T\mathcal{G} \rightarrow TM$ , called the *anchor* of  $A\mathcal{G}$ . This completes the construction of the *Lie algebroid of the Lie groupoid*  $\mathcal{G}$ .

**Examples 1.6.** (i) Consider  $\mathcal{G} = M \times M$ . The kernel of  $T(\alpha)$  is  $\beta^!(TM)$ , the pullback of  $TM$  to  $M \times M$  across  $\beta$ . Restricting this to the diagonal in  $M \times M$  we get  $A\mathcal{G} = TM$  as vector bundles. The right-invariant vector field corresponding to  $X \in \Gamma TM$  is  $\vec{X}(n, m) = X(n) \oplus 0_m$  and so the bracket on  $A\mathcal{G}$  is the standard bracket of vector fields on  $M$  and the anchor is the identity.

(ii) It is easy to see that a base-preserving morphism of Lie groupoids  $F: \mathcal{G} \rightarrow \mathcal{G}'$  induces a map  $A(F): A\mathcal{G} \rightarrow A\mathcal{G}'$  which is a morphism of Lie algebroids. In the case of  $\chi: \Pi(M) \rightarrow M \times M$  the kernel (that is, the set of elements of  $\Pi(M)$  which are mapped to identity elements of  $M \times M$ ), is the union of the fundamental groups of  $M$  and is fibre-wise discrete. It follows that  $A(\chi): A\Pi(M) \rightarrow TM$  is a diffeomorphism, usually identified with the identity.

(iii) The Lie algebroid  $A\Phi(E)$ , as a vector bundle, is an extension of  $TM$  by  $\text{End}(E)$ . Its sections are those linear differential operators  $D$ , of order  $\leq 1$ , for which there exists a vector field  $X$  on  $M$  such that

$$D(f\mu) = fD(\mu) + X(f)\mu,$$

for all  $f \in C^\infty(M)$  and  $\mu \in \Gamma E$ . These may be called *derivations* on  $E$ , and  $A\Phi(E)$  denoted by  $\mathfrak{D}(E)$ . The bracket on  $\mathfrak{D}(E)$  is the standard commutator bracket, and the anchor maps each  $D$  to its  $X$ .

We will not prove this, but it is easy to believe: the sections of  $\mathfrak{D}(E)$  are all covariant derivatives  $\nabla_X$  for all connections  $\nabla$  and all vector fields  $X$  on  $M$ , together with their differences, and  $\Phi(E)$  consists of all parallel translations, for all connections in  $E$  and all paths in  $M$ .

(iv) Let  $G \times M \rightarrow M$  be a smooth action of a Lie group  $G$  on a manifold  $M$  and let  $X \mapsto X_M$  be the infinitesimal action  $\mathfrak{g} \rightarrow \mathcal{X}(M)$ .

The Lie algebroid of  $G \ltimes M \rightrightarrows M$  is the trivial vector bundle  $M \times \mathfrak{g}$  with anchor  $M \times \mathfrak{g} \rightarrow TM$ ,  $(m, X) \mapsto X_M(m)$  and bracket

$$[V, W] = L_{V_M}(W) - L_{W_M}(V) + [V, W]^{\text{pt}}.$$

Here  $V$  and  $W$  are  $\mathfrak{g}$ -valued maps on  $M$ ; since  $\mathfrak{g}$  is a vector space we can take Lie derivatives of  $\mathfrak{g}$ -valued maps. The final term is the ‘pointwise bracket’ of  $V$  with  $W$  as maps into  $\mathfrak{g}$ .

## 2 Symplectic groupoids

The following definition is due to Weinstein [28]. Independent definitions were given by Karasëv [11] and Zakrzewski [30, 31].

**Definition 2.1.** A *symplectic groupoid* is a Lie groupoid  $\Sigma \rightrightarrows P$  together with a symplectic structure  $\omega$  on  $\Sigma$  such that the graph of multiplication

$$Gr = \{(hg, h, g) \mid \alpha h = \beta g\} \tag{7}$$

is Lagrangian in  $\overline{\Sigma} \times \Sigma \times \Sigma$ . The bar denotes reversal of the symplectic structure.

The following are the fundamental consequences:

**Theorem 2.2.** (i).  $1_P$  is Lagrangian in  $\Sigma$ .

(ii). Inversion is an antisymplectomorphism.

(iii). There is a unique Poisson structure on  $P$  such that  $\beta$  is a Poisson map.

(iv). There is a canonical isomorphism of Lie algebroids  $A\Sigma \cong T^*P$ .

The proof occupies the rest of the section. First the fundamental example.

### The cotangent groupoid of a Lie group

Consider any Lie group  $G$ . The tangent bundle  $TG$  has a Lie group structure with multiplication  $T(\kappa): TG \times TG \rightarrow TG$  where  $\kappa: G \times G \rightarrow G$  is the multiplication in  $G$ . Since the domain is a product manifold, we can differentiate in each variable separately and we get  $T(\kappa)(Y, X) = T(L_h)(X) + T(R_g)(Y)$  where  $Y \in T_h G$  and  $X \in T_g G$ . In practice we write  $Y \bullet X = T(\kappa)(Y, X)$ . With this structure,  $TG$  is the *tangent group* of  $G$ ,

It is possible to give the cotangent bundle  $T^*G$  a group structure in a similar way. However what we are about to define is distinct from the group structure.

Define a groupoid structure on  $T^*G$  with base  $\mathfrak{g}^*$  by

$$\begin{aligned}\tilde{\beta}(\varphi_g) &= \varphi \circ T_1(R_g), & \tilde{\alpha}(\varphi_g) &= \varphi \circ T_1(L_g), \\ \psi_h \cdot \varphi_g &= \psi \circ T(R_{g^{-1}}) = \varphi \circ T(L_{h^{-1}}).\end{aligned}\tag{8}$$

This structure is isomorphic to an action groupoid. Let  $G$  act on  $\mathfrak{g}^*$  by the coadjoint action  $g\theta := \theta \circ \text{Ad}_{g^{-1}}$  and form  $G \ltimes \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ . Then  $G \ltimes \mathfrak{g}^* \rightarrow T^*G$ ,  $(g, \theta) \mapsto \theta \circ T(L_{g^{-1}})$  is a groupoid isomorphism.

Whereas the group  $TG$  is isomorphic to the semi-direct product group  $G \ltimes \mathfrak{g}$  defined by the adjoint representation,  $T^*G$  is isomorphic to the action groupoid  $G \ltimes \mathfrak{g}^*$ . It is important to distinguish the two constructions.

The symplectic structure on  $T^*G$  is the standard symplectic structure for a cotangent bundle,  $\omega = d\lambda$  where  $\lambda$  is the 1-form on  $G$  defined on  $\xi \in T(T^*G)$  by

$$\langle \lambda, \xi \rangle = \langle \varphi, T(c)(\xi) \rangle$$

where  $\varphi \in T^*G$  is the base-point of  $\xi$  and  $c: T^*G \rightarrow G$  is the projection.

Further,  $\omega$  makes  $T^*G \rightrightarrows \mathfrak{g}^*$  a symplectic groupoid and the Poisson structure on  $\mathfrak{g}^*$  which arises from Theorem 2.2(iii) is the Lie–Poisson structure dual to  $\mathfrak{g}$ .

### Proof of Theorem 2.2

To see what Definition 2.1 means, we use the tangent groupoid of  $\Sigma$ . This construction applies to any Lie groupoid  $\mathcal{G} \rightrightarrows M$ . The *tangent groupoid* of  $\mathcal{G} \rightrightarrows M$  is the groupoid  $T\mathcal{G} \rightrightarrows TM$  obtained by applying the tangent functor to the structure of  $\mathcal{G} \rightrightarrows M$ . So the source of the

tangent groupoid is  $T(\alpha)$ , the identities are  $T(1)(x)$  for  $x \in TM$ , and the multiplication  $\eta \bullet \xi$  of two tangent vectors with  $T(\alpha)(\eta) = T(\beta)(\xi)$  is  $T(\kappa)(\eta, \xi)$  where  $\kappa: \mathcal{G} \times_M \mathcal{G} \rightarrow \mathcal{G}$  is the multiplication in  $\mathcal{G}$ . Because the tangent functor preserves diagrams, the groupoid axioms follow immediately. We write  $\xi^{-1}$  for the inverse of  $\xi \in T\mathcal{G}$ , and  $\tilde{0}_g$  for the zero element of  $T_g\mathcal{G}$ .

If  $\mathcal{G} = G$  is a group, then, as we saw above,  $TG$  is a group with  $\eta \bullet \xi = T(R_g)(\eta) + T(L_h)(\xi)$  for  $\eta \in T_h(G)$  and  $\xi \in T_g(G)$ . However no Lie group has a symplectic groupoid structure. For general Lie groupoids, the following result is very necessary.

**Proposition 2.3.** *For elements  $\xi_i \in T\mathcal{G}$ ,  $i = 1, \dots, 4$  with  $\xi_1, \xi_2 \in T_g\mathcal{G}$  and  $\xi_3, \xi_4 \in T_h\mathcal{G}$ , and with  $T(\alpha)(\xi_3) = T(\beta)(\xi_1)$  and  $T(\alpha)(\xi_4) = T(\beta)(\xi_2)$ , we have*

$$(\xi_4 + \xi_3) \bullet (\xi_2 + \xi_1) = \xi_4 \bullet \xi_2 + \xi_3 \bullet \xi_1 \quad (9)$$

PROOF. This is the statement that  $T(\kappa)$  is linear as a map of vector bundles  $T\mathcal{G} \times_{TM} T\mathcal{G} \rightarrow T\mathcal{G}$ .

To see this, observe that  $(\xi_3, \xi_1)$  and  $(\xi_4, \xi_2)$  are elements of  $T_h\mathcal{G} \times T_g\mathcal{G}$  and can therefore be added to give  $(\xi_4 + \xi_3, \xi_2 + \xi_1) \in T_h\mathcal{G} \times T_g\mathcal{G}$ .

The linearity then gives that

$$T(\kappa)(\xi_4 + \xi_3, \xi_2 + \xi_1) = T(\kappa)(\xi_4, \xi_2) + T(\kappa)(\xi_3, \xi_1)$$

which is (9). □

The source and target conditions ensure that each product and sum is defined and is in the domain of  $T(\kappa)$ .

Consider now a symplectic groupoid  $\Sigma \rightrightarrows P$ . The tangent to the graph (7) of the multiplication is the graph of the tangent multiplication :

$$T(Gr) = \{(\eta \bullet \xi, \eta, \xi) \mid T(\alpha)(\eta) = T(\beta)(\xi)\}. \quad (10)$$

Recall that a submanifold  $L$  of a symplectic manifold  $(S, \sigma)$  is *Lagrangian* if it is isotropic:  $\sigma(Y, X) = 0$  for all  $X, Y \in TL$ , and  $\dim L = \frac{1}{2} \dim S$ .

Applying the isotropy condition to the graph of the multiplication we have that

$$-\omega(\eta_1 \bullet \xi_1, \eta_2 \bullet \xi_2) + \omega(\eta_1, \eta_2) + \omega(\xi_1, \xi_2) = 0 \quad (11)$$

for all  $\xi_1, \xi_2, \eta_1, \eta_2 \in T\Sigma$  for which the multiplications are defined.

Set  $\xi_i = \eta_i = T(1)(x_i)$  in (11), where  $x_i \in TP$ . Then (11) becomes

$$-\omega(T(1)(x_1), T(1)(x_2)) + \omega(T(1)(x_1), T(1)(x_2)) + \omega(T(1)(x_1), T(1)(x_2)) = 0$$

so  $\omega(T(1)(x_1), T(1)(x_2)) = 0$  for all  $x_1, x_2 \in TP$ . This proves that  $1_P$  is isotropic in  $\Sigma$ . Next,

$$\dim(Gr) = \dim(\Sigma \times_P \Sigma) = 2 \dim \Sigma - \dim P$$

and this is to be  $\frac{1}{2} \dim(\Sigma \times \Sigma \times \Sigma)$  so  $\dim P = \frac{1}{2} \dim \Sigma$ . This completes the proof that  $1_P$  is Lagrangian in  $\Sigma$ .



Next consider any  $\xi_1, \xi_2$  and write  $x_i = T(\alpha)(\xi_i)$ . Then with  $\eta_i = \xi_i^{-1}$  in (11) we have

$$-\omega(T(1)(x_1), T(1)(x_2)) + \omega(\xi_1^{-1}, \xi_2^{-1}) + \omega(\xi_1, \xi_2) = 0$$

So

$$\omega(\xi_1^{-1}, \xi_2^{-1}) = -\omega(\xi_1, \xi_2), \quad (12)$$

which shows that inversion is antisymplectic.

The last two parts of Theorem 2.2 require more work.

Recall the map  $\omega^b: T\Sigma \rightarrow T^*\Sigma$  defined by  $\omega^b(\xi)(\eta) = -\omega(\xi, \eta)$ . What is  $\omega^b(\eta \bullet \xi)$ , where  $\eta \in T_h\Sigma$ ,  $\xi \in T_g\Sigma$  and  $T(\alpha)(\eta) = T(\beta)(\xi)$ ? It is an element of  $T_{hg}^*\Sigma$  so we pair it with an arbitrary element  $\zeta$  of  $T_{hg}\Sigma$ .

We can write  $\zeta = \zeta_2 \bullet \zeta_1$  where  $\zeta_2 \in T_h\Sigma$  and  $\zeta_1 \in T_g\Sigma$ . To see this, take any  $\zeta_1 \in T_g\Sigma$  with  $T(\alpha)(\zeta_1) = T(\alpha)(\zeta)$  and define  $\zeta_2 = \zeta \bullet \zeta_1^{-1}$ . This decomposition is not unique, of course.

Then, by (11),

$$\omega(\eta \bullet \xi, \zeta_2 \bullet \zeta_1) = \omega(\eta, \zeta_2) + \omega(\xi, \zeta_1).$$

Is the RHS well-defined? Suppose we also have  $\zeta = \zeta_4 \bullet \zeta_3$  where  $\zeta_4 \in T_h\Sigma$  and  $\zeta_3 \in T_g\Sigma$ . Then  $\zeta_3 \bullet \zeta_1^{-1} = \zeta_4^{-1} \bullet \zeta_2 \in T_{1_m}\Sigma$  where  $m = \alpha h = \beta g$ . Call this  $\nu$ . So  $\zeta_3 = \nu \bullet \zeta_1$  and  $\zeta_4 = \zeta_2 \bullet \nu^{-1}$ . Now

$$\omega(\eta \bullet \xi, \zeta_4 \bullet \zeta_3) = \omega(\eta, \zeta_2 \bullet \nu^{-1}) + \omega(\xi, \nu \bullet \zeta_1).$$

Consider the second term on the RHS first. Inserting an identity element and using (11), we have

$$\omega(\xi, \nu \bullet \zeta_1) = \omega(T(1)T(\beta)(\xi) \bullet \xi, \nu \bullet \zeta_1) = \omega(T(1)T(\beta)(\xi), \nu) + \omega(\xi, \zeta_1).$$

Now the first term:

$$\omega(\eta, \zeta_2 \bullet \nu^{-1}) = \omega(\eta \bullet T(1)(T(\alpha)(\eta)), \zeta_2 \bullet \nu^{-1}) = \omega(\eta, \zeta_2) + \omega(T(1)(T(\alpha)(\eta)), \nu^{-1}).$$

Lastly, using (12),

$$\omega(T(1)(T(\alpha)(\eta)), \nu^{-1}) = -\omega(T(1)(T(\alpha)(\eta)), \nu).$$

So

$$\omega(\eta \bullet \xi, \zeta) = \omega(\eta, \zeta_2) + \omega(\xi, \zeta_1) \quad (13)$$

is well-defined.

### The cotangent groupoid $T^*\Sigma$

Now we define a groupoid structure on  $T^*\Sigma$  so that (13) becomes

$$\omega^b(\eta \bullet \xi) = \omega^b(\eta) \bullet \omega^b(\xi). \quad (14)$$

In fact, we have already done most of the work. The base of  $T^*\Sigma$  will be  $A^*\Sigma$  and the source and target maps  $T^*\Sigma \rightarrow A^*\Sigma$  are

$$\langle \tilde{\beta}(\Phi), Y \rangle = \langle \Phi, T(R_g)(Y) \rangle, \quad \langle \tilde{\alpha}(\Phi), X \rangle = \langle \Phi, T(L_g)(X - T(1)(aX)) \rangle, \quad (15)$$

where  $\Phi \in T_g^* \Sigma$  and  $Y \in A_{\beta g} \Sigma$ ,  $X \in A_{\alpha g} \Sigma$ .

For  $\Phi \in T_g^* \Sigma$  and  $\Psi \in T_h^* \Sigma$  define  $\Psi \bullet \Phi \in T_{hg}^* \Sigma$  by

$$\langle \Psi \bullet \Phi, \eta \bullet \xi \rangle = \langle \Psi, \eta \rangle + \langle \Phi, \xi \rangle. \quad (16)$$

The proof that the RHS of (16) is well-defined needs precisely the condition that the source and target match, and follows the same pattern as the proof that (13) is well-defined. We leave the reader to check that when  $\Sigma = T^*G$  for a Lie group  $G$ , (16) reduces to (8).

To define the identity  $\tilde{1}_\varphi \in T_{1_m}^* \Sigma$  corresponding to  $\varphi \in A_m^* \Sigma$ , observe that every element  $\xi$  of  $T_{1_m} \Sigma$  can be written uniquely in the form  $T(1)(x) + X$  where  $x \in T_m M$  and  $X \in A_m \Sigma$ . We can therefore define

$$\langle \tilde{1}_\varphi, T(1)(x) + X \rangle = \langle \varphi, X \rangle \quad (17)$$

and it is straightforward to check that  $\tilde{1}_\varphi$  is indeed an identity for the multiplication and that this structure makes  $T^* \Sigma \rightrightarrows A^* \Sigma$  a groupoid with inverses

$$\langle \Phi^{-1}, \xi^{-1} \rangle = -\langle \Phi, \xi \rangle. \quad (18)$$

We now prove that  $\omega^b: T\Sigma \rightarrow T^* \Sigma$  is a morphism of groupoids,

$$\begin{array}{ccc} T\Sigma & \xrightarrow{\omega^b} & T^* \Sigma \\ \Downarrow & & \Downarrow \\ TP & \xrightarrow{w} & A^* \Sigma \end{array} \quad (19)$$

over some map  $w$ . To find  $w$ , take the composite of  $T(1)$ , followed by  $\omega^b$ , followed by  $\tilde{\beta}$  (or  $\tilde{\alpha}$ ). This gives

$$w(x) = -\iota_{T(1)(x)} \omega|_{A\Sigma}.$$

We must now show that  $\tilde{\beta} \circ \omega^b = w \circ T(\beta)$ . Take  $\xi \in T_g \Sigma$ . Then

$$\langle \tilde{\beta}(\omega^b(\xi)), Y \rangle = \langle \omega^b(\xi), T(R_g)(Y) \rangle = -\omega(\xi, T(R_g)(Y)).$$

On the other hand,

$$\langle w(T(\beta)(\xi)), Y \rangle = -\omega(T(1)(T(\beta)(\xi)), Y).$$

To prove the equality of these, we need to express right-translations in terms of the tangent groupoid multiplication.

**Lemma 2.4.** For  $X \in A_m \Sigma$  and  $g \in \Sigma$  with  $\beta(g) = m$ ,

$$X \bullet \tilde{0}_g = T(R_g)(X). \quad (20)$$

PROOF. Write  $X$  as the derivative at  $t = 0$  of a curve  $h_t \in \Sigma_m$  with  $h_0 = 1_m$ . One can likewise write  $\tilde{0}_g$  as the derivative of the curve constant at  $g$ . Then  $X \bullet \tilde{0}_g$  is the derivative of the curve  $h_t g$  and this is  $T(R_g)(X)$ .

We now complete the proof that  $\tilde{\beta} \circ \omega^b = w \circ T(\beta)$ . We must show that

$$\omega(\xi, T(R_g)(Y)) = \omega(T(1)(T(\beta)(\xi)), Y).$$

Using (11) and (20) the LHS is equal to

$$\omega(T(1)T(\beta)(\xi) \bullet \xi, Y \bullet \tilde{0}_g) = \omega(T(1)T(\beta)(\xi), Y) + \omega(\xi, \tilde{0}_g)$$

and  $\omega(\xi, \tilde{0}_g)$  is zero, since  $\tilde{0}_g$  is the zero of  $T_g\Sigma$ .

The proof that  $\tilde{\alpha} \circ \omega^b = w \circ T(\alpha)$  is similar.  $\square$

Finally, the morphism property itself is (13). Note that  $w$  is an isomorphism of vector bundles since  $\omega^b$  is.

### Poisson structure on $P$

Consider Figure 1(a). This diagram and (b) are not diagrams of morphisms (as in (19)) but are single mathematical objects, part of the structure of which is indicated by the arrows. (See §5.)

$$\begin{array}{ccc} T\Sigma & \xrightarrow{\tilde{q}} & \Sigma \\ \Downarrow & & \Downarrow \\ TP & \xrightarrow{p} & P \end{array} \quad \begin{array}{ccc} T^*\Sigma & \xrightarrow{\tilde{c}} & \Sigma \\ \Downarrow & & \Downarrow \\ A^*\Sigma & \xrightarrow{q} & P \end{array}$$

(a) (b)

Figure 1

Recall that the Lie algebroid of  $\Sigma$ , as a vector bundle, is the intersection of the kernel of the bundle projection  $\tilde{q}$ , which is the restriction of  $T\Sigma$  to the identity elements, and the kernel of  $T(\alpha)$ , which is the vector bundle tangent to the source fibres. The bracket on  $\Gamma A\Sigma$  is given by associating to a section  $X$  of  $A\Sigma$  a right-invariant vector field  $\vec{X}$ , and showing that the bracket of right-invariant vector fields is right-invariant. We now apply this process to the cotangent structure in Figure 1(b).

First find the kernel of  $\tilde{\alpha}$ . To do this we use the following lemma.

**Lemma 2.5.** *Let  $\eta \in T_h\Sigma$  and  $\xi \in T_g\Sigma$  have  $T(\alpha)(\eta) = 0 = T(\beta)(\xi)$ . Then*

$$\eta \bullet \xi = T(L_h)(\xi) + T(R_g)(\eta).$$

PROOF. In this case  $\eta$  is tangent to  $\alpha^{-1}(m)$  and  $\xi$  is tangent to  $\beta^{-1}(m)$  where  $m = \alpha h = \beta g$ , and so  $\eta \bullet \xi$  can be calculated from the restriction of multiplication to  $\alpha^{-1}(m) \times \beta^{-1}(m) \rightarrow \Sigma$ , which does not involve a pullback.  $\square$

Now take  $X \in A\Sigma$ . Then, applying the lemma, we have

$$X \bullet (T(1)(aX) - X) = X + (T(1)(aX) - X) = T(1)(aX)$$

and so

$$X^{-1} = T(1)(aX) - X.$$

So we can rewrite the definition of  $\tilde{\alpha}$  as

$$\langle \tilde{\alpha}(\Phi), X \rangle = -\langle \Phi, T(L_g)(X^{-1}) \rangle,$$

and the kernel of  $\tilde{\alpha}$  is therefore the annihilator of  $T^\beta \Sigma$ .

Suppose  $T(\beta)(\xi) = 0$ , where  $\xi \in T_g \Sigma$ , Then  $T(\alpha)(\xi^{-1}) = 0$ . So  $\xi^{-1} = T(R_{g^{-1}})(X)$  for some  $X \in A\Sigma$ . That is,  $\xi^{-1} = X \bullet \tilde{0}_{g^{-1}}$ . Thence  $\xi = \tilde{0}_g \bullet X^{-1}$ .

Clearly all pullbacks  $\beta^* \mu$ , for  $\mu \in T^*P$ , annihilate  $T^\beta \Sigma$ . Conversely, suppose that  $\Phi \in T_{1_m}^* \Sigma$  annihilates  $T^\beta \Sigma$ . Then  $\langle \Phi, X \rangle = \langle \Phi, T(1)(aX) \rangle$  for all  $X \in A_m \Sigma$ . Define  $\mu \in T_m^* P$  by  $\langle \mu, x \rangle = \langle \Phi, T(1)(x) \rangle$  for  $x \in T_m P$ . Then, for all  $T(1)(x) + X \in T_{1_m} \Sigma$ ,

$$\langle \Phi, T(1)(x) + X \rangle = \langle \Phi, T(1)(x) + T(1)(aX) \rangle = \langle \mu, x + aX \rangle = \langle \beta^* \mu, T(1)(x) + X \rangle,$$

so  $\Phi = \beta^* \mu$ . Thus the bundle for Figure 1(b) which corresponds to  $A\Sigma$  in (a), is  $T^*P$ .

Now take  $\mu \in \Omega^1(P)$ . We define  $\vec{\mu} \in \Omega^1(\Sigma)$  by

$$\vec{\mu}(g) = \mu(\beta g) \bullet \hat{0}_g,$$

where  $\hat{0}_g$  is the zero element of  $T_g^* G$ . In fact  $\mu(\beta g) \bullet \hat{0}_g$  is merely the pullback of  $\mu(\beta g)$  to  $g$ .

Take  $\mu \in T_m^* P$  and  $g$  with  $\beta g = m$ . Then

$$\langle \mu \bullet \hat{0}_g, \eta \bullet \xi \rangle = \langle \mu, \eta \rangle + 0$$

where  $\eta = T(1)(y) + Y$  and  $T(\beta)(\xi) = T(\alpha)(\eta) = y$ . Now, regarding  $\mu$  as in  $T_{1_m}^* \Sigma$ , we have

$$\langle \mu, \eta \rangle = \langle \mu, T(\beta)(\eta) \rangle = \langle \mu, y + aY \rangle.$$

On the other hand,

$$\langle \beta_g^* \mu, \eta \bullet \xi \rangle = \langle \mu, T(\beta)(\eta) \rangle = \langle \mu, y + aY \rangle.$$

We can now define a bracket on  $\Omega^1(P)$  by

$$\overrightarrow{[\mu_1, \mu_2]} = [\vec{\mu}_1, \vec{\mu}_2], \quad (21)$$

where the bracket on the right is the bracket on  $\Omega^1(\Sigma)$  transported via  $\omega^b$  from  $\mathcal{X}(\Sigma)$ . The bracket on  $\Omega^1(P)$  is therefore skew-symmetric and satisfies the Jacobi identity. Note that (21) can also be written as

$$\beta^*[\mu_1, \mu_2] = [\beta^* \mu_1, \beta^* \mu_2], \quad (22)$$

Regarding  $A\Sigma \subseteq T\Sigma$  and  $T^*P \subseteq T^*\Sigma$  as the intersections of the relevant kernels, it follows from the fact that  $\omega^b$  is an isomorphism of vector bundles, that the restriction to  $A\Sigma \rightarrow T^*P$  is also. Denote this restriction temporarily by  $r$ . Then,

$$r = -w^*. \quad (23)$$

To see this, take  $X \in A\Sigma$  and  $y \in TP$ . Then

$$\langle w^*(X), y \rangle = \langle w(y), X \rangle = -\omega(T(1)(y), X),$$

whereas

$$\langle r(X), y \rangle = \langle \omega^b(X), T(1)(y) \rangle = -\omega(X, T(1)(y)).$$

The brackets in  $\Gamma A\Sigma$  and  $\Gamma T^*P$  are defined in terms of those on  $\mathcal{X}(\Sigma)$  and  $\Omega^1(\Sigma)$  by analogous processes and it follows that  $r$  is an isomorphism of Lie algebroids, to  $T^*P$  with the bracket defined by (21) and the anchor  $a \circ r^{-1}$ , where  $a$  is the anchor of  $A\Sigma$ .

**Lemma 2.6.**

$$(a \circ r^{-1})^* = -a \circ r^{-1}.$$

PROOF. We first prove that  $w \circ a = a^* \circ r$ . For  $X, Y \in A\Sigma$ ,

$$\langle (w \circ a)(X), Y \rangle = \omega(T(1)(aX), Y)$$

and

$$\langle (a^* \circ r)(X), Y \rangle = \langle r(X), a(Y) \rangle = -\langle w^*(X), a(Y) \rangle = -\langle w(aY), X \rangle = -\omega(T(1)(aY), X).$$

We know that  $T(1)(aX) = X + X^{-1}$  and  $T(1)(aY) = Y + Y^{-1}$ . So we have

$$\langle (w \circ a)(X), Y \rangle = \omega(X, Y) + \omega(X^{-1}, Y) \text{ and } \langle (a^* \circ r)(X), Y \rangle = \omega(X, Y) + \omega(X, Y^{-1}).$$

We must show that  $\omega(X^{-1}, Y) = -\omega(X, Y^{-1}) = 0$ .

Take  $X \in A\Sigma$ . Since  $r$  maps  $X$  to  $r(X) \in T^*P$ , it follows that  $\omega^b$  maps  $\vec{X}$  to  $\overrightarrow{r(X)}$ ; that is,

$$\omega^b(\vec{X}) = \beta^*(r(X)) = -\beta^*(w^*(X)). \quad (24)$$

So for any  $\eta \in T\Sigma$ ,  $\omega(\vec{X}, \eta) = -\langle \beta^*(w^*(X)), \eta \rangle = -\langle w^*(X), T(\beta)(\eta) \rangle$ . In particular if  $T(\beta)(\eta) = 0$  then  $\omega(\vec{X}, \eta) = 0$ . This completes the proof that  $w \circ a = a^* \circ r$ .

Rewrite this as  $a \circ r^{-1} = w^{-1} \circ a^*$ . Now

$$(a \circ r^{-1})^* = (w^{-1} \circ a^*)^* = a \circ (w^*)^{-1} = -a \circ r^{-1},$$

as required. □

Note from the proof that we have also shown that:

**Proposition 2.7.** *For  $\xi, \eta \in T\Sigma$ , if  $T(\alpha)(\xi) = 0$  and  $T(\beta)(\eta) = 0$  then  $\omega(\xi, \eta) = 0$ .*

Now define a Poisson structure  $\pi$  on  $P$  by

$$\{f_1, f_2\} = \langle df_2, (a \circ r^{-1})(df_1) \rangle; \quad (25)$$

that is,  $\pi^\# = a \circ r^{-1}$ . That  $\pi^\#$  is skew-symmetric has just been proved, and the Jacobi identity follows from the isomorphism of  $T^*P$  with  $A\Sigma$ . So  $\pi$  is a Poisson structure on  $P$  and, by (22),  $\beta$  is a Poisson map. Since  $\beta$  is a surjective submersion, it is the only such Poisson structure.

This completes the proof of (iii) and (iv) of Theorem 2.2.

**Proposition 2.8.** *The diffeomorphism  $-w: TP \rightarrow A^*\Sigma$  is a Poisson map from the tangent lift Poisson structure on  $TP$  to the Poisson structure dual to the Lie algebroid structure on  $A\Sigma$ .*

The *tangent lift Poisson structure on  $TP$*  can be defined as the Poisson structure which is dual to the Lie algebroid  $T^*P$ . It was originally defined by T. Courant [5] directly in terms of functions on  $TP$ . For  $f \in C^\infty(P)$  denote the function  $TP \rightarrow \mathbb{R}$  corresponding to  $df: P \rightarrow T^*P$  by  $\ell_{df}$ . Then it is sufficient to define the Poisson brackets for all  $\ell_{df}$  and all  $p^*f$ , where  $p: TP \rightarrow P$  is the bundle projection. As in [5], we define:

$$\{\ell_{df_1}, \ell_{df_2}\} = \ell_{d\{f_1, f_2\}}, \quad \{\ell_{df_1}, p^*f_2\} = p^*\{f_1, f_2\}, \quad \{p^*f_1, p^*f_2\} = 0. \quad (26)$$

Proposition 2.8 now follows from the following general result. The proof is a simple exercise.

**Proposition 2.9.** *Let  $A$  and  $B$  be Lie algebroids on the same base  $M$ , and let  $\varphi: A \rightarrow B$  be a morphism of vector bundles over  $M$ . Then  $\varphi$  is a morphism of Lie algebroids if and only if  $\varphi^*: B^* \rightarrow A^*$  is a Poisson map.*

### 3 Midword

We have given the proof that the base manifold of a symplectic groupoid has a Poisson structure such that the target is a Poisson map, and that the cotangent Lie algebroid of this Poisson structure is canonically isomorphic to the Lie algebroid of the Lie groupoid. This is a very striking result. It shows, in particular, that the Poisson structure on the base manifold determines the symplectic groupoid up to local isomorphism.

Theorem 2.2 is the apotheosis of a classical question, as to whether a given Poisson manifold can be realized as the quotient of a symplectic manifold. In the example  $T^*G \rightrightarrows \mathfrak{g}^*$  for  $G$  a Lie group, the Poisson manifold is the quotient of  $T^*G$  under an action of  $G$ . In the general case of Theorem 2.2, the symplectic manifold  $\Sigma$  is quotiented to its base  $P$  by the surjective submersion  $\beta: \Sigma \rightarrow P$ . This may be regarded as the quotient over the action of the equivalence relation  $\Sigma \times_\beta \Sigma$  on  $\Sigma$ .

The classical question is the ‘realizability problem’: given a Poisson manifold  $P$ , is there a symplectic manifold  $M$  and a surjective submersion  $M \rightarrow P$  which is a Poisson map? In fact, under mild conditions, such a map can be modified to give a symplectic groupoid structure.

The answer to the question — *given a Poisson manifold  $P$  is there a symplectic groupoid  $\Sigma \rightrightarrows P$ ?* — was provided by Crainic and Fernandes [6, 7], building on work of Cattaneo and Felder [3].

The modern theory of symplectic realizations was introduced by Weinstein [28, 4] in order to provide a route for the quantization of Poisson manifolds.

In this article, we are concerned with the structures arising from symplectic groupoids. These are, I believe, most easily understood by studying Poisson groupoids, of which symplectic groupoids are a particular case. Poisson groupoids were introduced by Weinstein [29] and gave rise to the theory of Lie bialgebroids. It is to some aspects of this theory that we now turn.

## 4 Poisson groupoids

Let  $P$  be a Poisson manifold. A closed submanifold  $C \subseteq P$  is *coisotropic* if the Poisson anchor  $\pi^\# : T^*P \rightarrow TP$ , when restricted to  $(TC)^\circ$ , goes into  $TC$ . Here  $(TC)^\circ$  is the *annihilator of  $TC$  in  $T^*P$* ; it is isomorphic to the conormal bundle  $T_C^*P/TC$ .

In terms of the bracket of functions, a closed submanifold  $C$  is coisotropic in  $P$  if, whenever  $f, g \in C^\infty(P)$  vanish on  $C$ , their bracket  $\{f, g\}$  does also.

Coisotropic submanifolds play the role that in symplectic geometry is played by Lagrangian submanifolds. In particular, a map  $\varphi : P \rightarrow Q$  of Poisson manifolds is a Poisson map if and only if the graph of  $\varphi$  is a coisotropic submanifold of  $\overline{Q} \times P$ .

**Definition 4.1** ([29]). A *Poisson groupoid* is a Lie groupoid  $\mathcal{G} \rightrightarrows P$  together with a Poisson structure on  $\mathcal{G}$  such that the graph of multiplication

$$Gr = \{(hg, h, g) \mid \alpha h = \beta g\} \quad (27)$$

is coisotropic in  $\overline{\mathcal{G}} \times \mathcal{G} \times \mathcal{G}$ .

Using (10),  $T(Gr)^\circ$  consists of triples  $(\Theta, \Psi, \Phi)$  of elements of  $T^*\mathcal{G}$  such that

$$\Theta(\eta \bullet \xi) + \Psi(\eta) + \Phi(\xi) = 0 \quad (28)$$

for all  $\eta, \xi \in T\mathcal{G}$  such that  $T(\alpha)(\eta) = T(\beta)(\xi)$ . We claim that this implies  $\Theta = -\Psi \bullet \Phi$ . First we must show that  $\tilde{\alpha}(\Psi) = \tilde{\beta}(\Phi)$ . Let  $h, g \in \mathcal{G}$  be the base elements of  $\eta, \xi$  and write  $m = \beta(g)$ . For  $Y \in A_m\mathcal{G}$  we have

$$(\tilde{\beta}(\Phi))(Y) = \Phi(T(R_g)(Y)) = \Phi(Y \bullet \tilde{0}_g).$$

From (28) we now get

$$\Phi(Y \bullet \tilde{0}_g) = \Theta(\tilde{0}_h \bullet \tilde{0}_g) - \Psi(\tilde{0}_h \bullet Y^{-1}) = -\Psi(T(L_h)(Y^{-1}))$$

However  $Y^{-1} = T(1)(aY) - Y$ , since  $Y \in A\mathcal{G}$ , and so we have

$$(\tilde{\beta}(\Phi))(Y) = \Psi(T(L_h)(Y - T(1)(aY))) = (\tilde{\alpha}(\Psi))(Y).$$

Now (28) can be written

$$T(Gr)^\circ = \{(-\Psi \bullet \Phi, \Psi, \Phi) \mid \tilde{\alpha}(\Psi) = \tilde{\beta}(\Phi)\}. \quad (29)$$

The coisotropy condition now is that

$$\pi^\#(\Psi \bullet \Phi) = \pi^\#(\Psi) \bullet \pi^\#(\Phi) \quad (30)$$

for  $\Psi, \Phi \in T^*\mathcal{G}$  with  $\tilde{\alpha}(\Psi) = \tilde{\beta}(\Phi)$ . This is evidently a morphism condition on  $\pi^\#$ . Denote the base map  $A^*\mathcal{G} \rightarrow TP$  temporarily by  $b$ . Then, for  $\varphi \in A^*\mathcal{G}$ ,

$$\pi^\#(\tilde{1}_\varphi) = T(1)(b(\varphi)). \quad (31)$$

From this it follows that  $1_P$  is coisotropic in  $\mathcal{G}$ . For if  $\Phi \in T_{1_m}^*\mathcal{G}$  annihilates  $T_{1_m}(1_P)$  then, by (17),  $\Phi = \tilde{1}_\varphi$  for some  $\varphi \in A^*\mathcal{G}$  and (31) is then the coisotropy condition for  $1_P$ .

That  $1_P$  is coisotropic in  $\mathcal{G}$  implies, as shown by Weinstein [29], that  $A^*\mathcal{G}$  inherits a Lie algebroid structure from  $T^*\mathcal{G}$ , the anchor of which is the restriction of  $\pi^\#$ ; that is,  $b$ . We therefore write  $a_*$  for  $b$  in what follows.

To define the bracket we must verify two conditions, namely that

- if  $\Phi, \Psi \in \Gamma T^*\mathcal{G}$  have  $\Phi|_P, \Psi|_P \in \Gamma A^*\mathcal{G}$  then  $[\Phi, \Psi]|_P \in \Gamma A^*\mathcal{G}$  also;
- if  $\Phi, \Psi \in \Gamma T^*\mathcal{G}$  have  $\Phi|_P = 0$  and  $\Psi|_P \in \Gamma A^*\mathcal{G}$ , then  $[\Phi, \Psi]|_P = 0$ .

For the first, take any  $\xi \in \mathcal{X}(P)$  and extend it to a vector field on  $\mathcal{G}$ , also denoted  $\xi$ . Take  $\Phi, \Psi \in \Gamma T^*\mathcal{G}$  such that  $\Phi|_P, \Psi|_P$  are sections of  $(TP)^\circ$ . From (3) we have

$$\langle [\Phi, \Psi], \xi \rangle = \langle \mathcal{L}_{\Phi^\#}(\Psi), \xi \rangle - \langle \mathcal{L}_{\Psi^\#}(\Phi), \xi \rangle - \langle d(\pi(\Phi, \Psi)), \xi \rangle. \quad (32)$$

The first term is

$$\langle \mathcal{L}_{\Phi^\#}(\Psi), \xi \rangle = \mathcal{L}_{\Phi^\#}(\langle \Psi, \xi \rangle) - \langle \Psi, [\Phi^\#, \xi] \rangle. \quad (33)$$

On the RHS, the first term is zero on  $P$  because  $\langle \Psi, \xi \rangle$  is constant on  $P$  and  $\Phi^\#$  is tangent to  $P$ . In the second term, both  $\Phi^\#$  and  $\xi$  are tangent to  $P$ , and so their Lie bracket is also; hence the pairing with  $\Psi$  is zero on  $P$ .

The second term of (32) is likewise zero on  $P$ . The third term is  $-\xi(\pi(\Phi, \Psi))$  and  $\pi(\Phi, \Psi) = \langle \Psi, \Phi^\# \rangle$  vanishes on  $P$  since  $\Phi^\#$  is tangent to  $P$ . Since  $\xi \in \mathcal{X}(P)$  was arbitrary,  $[\Phi, \Psi]$  is a section of  $(TP)^\circ$ .

To verify the second itemized condition, take  $\Phi, \Psi \in \Gamma T^*\mathcal{G}$  with  $\Phi|_P = 0$  and  $\Psi|_P \in \Gamma(TP)^\circ$ . Let  $\xi$  be any vector field on  $P$ .

Now in (33), the first term on the RHS is zero on  $P$  because  $\Phi^\#$  is zero on  $P$ . The second term,  $-\langle \Psi, [\Phi^\#, \xi] \rangle$ , may not be zero; see below.

In the corresponding equation for  $\langle \mathcal{L}_{\Psi^\#}(\Phi), \xi \rangle$ , the first term on the RHS is zero on  $P$  because the bracket is zero on  $P$  and  $\Psi^\#$  is tangent to  $P$ , by coisotropy. The second term is zero on  $P$  because  $\Phi$  is zero on  $P$ .

The third term is equal to  $\mathcal{L}_\xi(\langle \Psi, \Phi^\# \rangle) = \langle \mathcal{L}_\xi(\Psi), \Phi^\# \rangle + \langle \Psi, [\xi, \Phi^\#] \rangle$ . The first term on the RHS is zero on  $P$  because  $\Phi^\#$  is so. The second term may be nonzero, and cancels with the term above.

Since  $\xi$  was any vector field on  $\mathcal{G}$ , this proves that  $[\Phi, \Psi]$  is zero on  $P$ .

This shows that  $(TP)^\circ$  is a Lie subalgebroid of  $T^*\mathcal{G}$ . It is also possible to regard the Lie algebroid structure on  $A^*\mathcal{G}$  as a quotient of the Lie algebroid  $T^*\mathcal{G}$ .

We have now proved part of the following theorem, which should be compared with Theorem 2.2.

**Theorem 4.2.** *Let  $\mathcal{G} \rightrightarrows P$  be a Poisson groupoid. Then:*

- The Poisson anchor  $\pi^\#: T^*\mathcal{G} \rightarrow T\mathcal{G}$  is a morphism of Lie groupoids with base map denoted  $a_*: A^*\mathcal{G} \rightarrow TP$ .



(ii).  $1_P$  is coisotropic in  $\mathcal{G}$ .

(iii). The Lie algebroid structure on  $T^*\mathcal{G}$  induces a Lie algebroid structure on the dual bundle  $A^*\mathcal{G}$  for which the anchor is  $a_*$ .

(iv). Inversion is an antiPoisson map.

(v). There is a unique Poisson structure on  $P$  such that  $\beta$  is a Poisson map.

The chief difference between Theorem 4.2 and Theorem 2.2 concerns the Lie algebroid structure on  $A^*\mathcal{G}$ . In 2.2(iv) there is a natural isomorphism of Lie algebroids  $A\Sigma \cong T^*P$ . The dual of this,  $TP \rightarrow A^*\Sigma$ , may be used to put a Lie algebroid structure on  $A^*\Sigma$ , though there is little reason to do this if one is working only with symplectic groupoids.

The proof of Theorem 4.2(iv) is similar to that for symplectic groupoids. Note however that whereas inversion  $\mathcal{G} \rightarrow \mathcal{G}$  is an antiPoisson diffeomorphism, the groupoid inversion  $T^*\mathcal{G} \rightarrow T^*\mathcal{G}$  is a Lie algebroid isomorphism, without change of sign. This confirms, in a small way, the naturality of working with the cotangent structures.

The Poisson structure on  $P$  is defined by  $\pi_P^\# = a_* \circ a^* = -a \circ a_*^*$ . It follows that  $\beta: \mathcal{G} \rightarrow P$  is a Poisson map.

We need to explicate further the relationship between the Lie algebroid structures on  $A\mathcal{G}$  and  $A^*\mathcal{G}$ ; in particular we want to express the relationship between  $A\mathcal{G}$  and  $A^*\mathcal{G}$  without reference to the underlying Poisson groupoid.

With any Lie algebroid  $A$  on base manifold  $M$  there is a cochain complex

$$C^\infty(M) \xrightarrow{d} \Gamma(A^*) \xrightarrow{d} \Gamma\Lambda^2(A^*) \xrightarrow{d} \Gamma\Lambda^3(A^*) \xrightarrow{d} \dots$$

defined by the natural extension of the coboundary operator for de Rham cohomology and for Lie algebra cohomology.

**Theorem 4.3.** *Let  $\mathcal{G} \rightrightarrows P$  be a Poisson groupoid. Then for the coboundary operator  $d_*$  of the Lie algebroid  $A^*\mathcal{G}$  defined above we have, for  $X, Y \in \Gamma A\mathcal{G}$ ,*

$$d_*[X, Y] = [X, d_*Y] + [d_*X, Y]. \quad (34)$$

We will not give the proof here; see [20] or [18, §12.1].

On the RHS we have the bracket of sections of  $A\mathcal{G}$  with sections of  $\Gamma\Lambda^2(A\mathcal{G})$ . This bracket is defined in terms of decomposable elements by

$$[X, Y \wedge Z] = [X, Y] \wedge Z + Y \wedge [X, Z].$$

and  $[X, \eta] = -[\eta, X]$  for  $X \in \Gamma A\mathcal{G}$  and  $\eta \in \Gamma\Lambda^2(A\mathcal{G})$ . The bracket on  $\Gamma A\mathcal{G}$  may be extended to the exterior algebra  $\Gamma\Lambda(A\mathcal{G})$ ; this is the *Schouten* or *Gerstenhaber bracket* for  $A\mathcal{G}$ . Details and references are given in [18, §7.5].

**Definition 4.4.** Let  $A$  be a Lie algebroid on  $M$  and suppose that  $A^*$  has a Lie algebroid structure. Then  $(A, A^*)$  is a *Lie bialgebroid* if (34) holds for all  $X, Y \in \Gamma A$ .

Lie bialgebroids were defined by Ping Xu and the author in [20], following the construction of the Lie algebroid structure on the dual by Weinstein [29]. Equation (34) was immediately shown by Kosmann-Schwarzbach [12] to extend to elements of arbitrary degree of  $\Gamma\Lambda(A)$ ; therefore  $d_*$  is a derivation of the Schouten bracket on  $\Gamma\Lambda(A)$ . Equation (34) implies its dual form,  $d[\varphi, \psi]_* = [\varphi, d\psi]_* + [d\varphi, \psi]_*$ , where  $\varphi, \psi \in \Gamma A^*$  and  $[\ , \ ]_*$  is the bracket on  $\Gamma A^*$ .

Equation (34) is thus in many respects well-understood, but it remains a very nonlinear equation. We will now show how the relationship between the Lie algebroid structures on  $A\mathcal{G}$  and  $A^*\mathcal{G}$  may be understood in a ‘diagrammatic’ way.

## 5 The diagrammatic approach

We work with a given Poisson groupoid  $\mathcal{G} \rightrightarrows P$ . The key to the diagrammatic approach is to work with the cotangent bundle  $T^*\mathcal{G}$  rather than  $\mathcal{G}$  itself. Since  $\mathcal{G}$  is a Poisson manifold,  $T^*\mathcal{G}$  is a Lie algebroid, and since  $\mathcal{G}$  is a Lie groupoid,  $T^*\mathcal{G}$  has a Lie groupoid structure. These structures are shown in Figure 2(a).

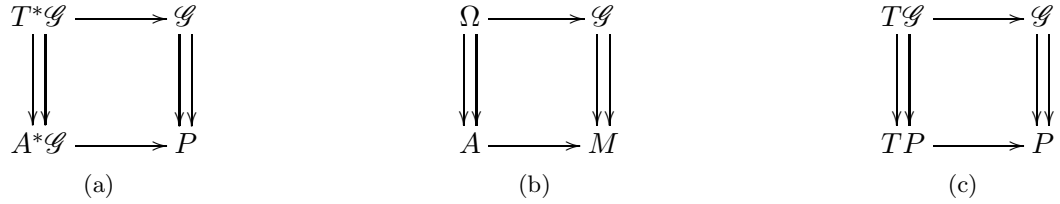


Figure 2.

For any Lie groupoid  $\mathcal{G} \rightrightarrows P$ , it is straightforward to show that  $T^*\mathcal{G}$  is a  $\mathcal{VB}$ -groupoid in the following sense.

**Definition 5.1** ([24]). A  $\mathcal{VB}$ -groupoid  $(\Omega; \mathcal{G}, A; M)$  is a structure as shown in Figure 2(b) in which  $\Omega$  is a vector bundle over  $\mathcal{G}$ , which is a Lie groupoid over  $M$ , and  $\Omega$  is also a Lie groupoid over  $A$ , which is a vector bundle over  $M$ , subject to the condition that the structure maps of the groupoid structure (source, target, identity, multiplication, inversion) are vector bundle morphisms, and the ‘double source map’  $\Omega \rightarrow \mathcal{G} \times_M A$  formed from the bundle projection and the source on  $\Omega$ , is a surjective submersion.

According to the duality for  $\mathcal{VB}$ -groupoids introduced by Pradines [24], the  $\mathcal{VB}$ -groupoid  $T^*\mathcal{G}$  is dual to the  $\mathcal{VB}$ -groupoid  $T\mathcal{G}$  shown in Figure 2(c), the tangent prolongation of  $\mathcal{G} \rightrightarrows P$ .

Each diagram in Figure 2 denotes the full set of structures just described, and constitutes a single mathematical object; they should not be read, for example, as showing a morphism of groupoids.

For  $\mathcal{G} \rightrightarrows P$  a Poisson groupoid, each horizontal structure in Figure 2(a) is a Lie algebroid. These are compatible with the vertical groupoid structures in the following sense.

**Definition 5.2** ([16]). An  $\mathcal{LA}$ -groupoid  $(\Omega; \mathcal{G}, A; M)$  is a structure as shown in Figure 2(b) in which  $\Omega$  is a Lie algebroid over  $\mathcal{G}$ , which is a Lie groupoid over  $M$ , and  $\Omega$  is also a Lie groupoid

over  $A$ , which is a Lie algebroid over  $M$ , subject to the condition that the structure maps of the groupoid structure (source, target, identity, multiplication, inversion) are Lie algebroid morphisms, and the ‘double source map’  $\Omega \rightarrow \mathcal{G} \times_M A$  formed from the bundle projection and the source on  $\Omega$ , is a surjective submersion.

We proved in §4 that the anchor  $\pi^\# : T^*\mathcal{G} \rightarrow T\mathcal{G}$  is a morphism of Lie groupoids; that result may equally be formulated as stating that each of the groupoid structure maps preserves the anchors. It is necessary to also prove that the groupoid structure maps preserve the brackets; for this proof, see [16]. Thus  $T^*\mathcal{G}$ , for  $\mathcal{G} \rightrightarrows P$  a Poisson groupoid, is an  $\mathcal{LA}$ -groupoid. This has important ramifications: one may differentiate an  $\mathcal{LA}$ -groupoid, and one may seek to integrate it.

Consider first the result of differentiating the  $\mathcal{LA}$ -groupoid in Figure 2(b). Applying the Lie functor to the vertical groupoid structures we obtain a structure as shown in Figure 3(a). Here each arrow represents a vector bundle structure; altogether these make  $A\Omega$  a double vector bundle, as defined by Pradines [22, 23].

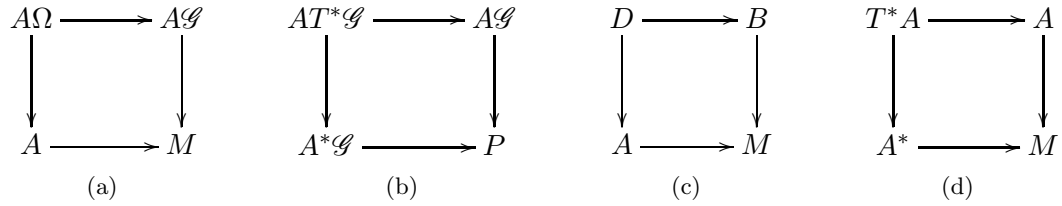


Figure 3.

The vertical structures are evidently Lie algebroids. It is also true, though not so very quick to prove [17], that the horizontal structures inherit Lie algebroid structures by a process which is an extension of the construction of the tangent prolongation of a Lie algebroid. Thus all four sides of Figure 3(a) have Lie algebroid structures.

Applying this to the cotangent  $\mathcal{LA}$ -groupoid of a Poisson groupoid, Figure 2(a), we obtain Figure 3(b). Here there is a canonical isomorphism  $AT^*\mathcal{G} \cong T^*A\mathcal{G}$  which preserves the Lie algebroid structures [20]. Thus the Lie algebroid structure on  $AT^*\mathcal{G} \rightarrow A\mathcal{G}$  may be obtained as the cotangent Lie algebroid structure of the Poisson structure on  $A\mathcal{G}$  which is dual to the Lie algebroid structure on  $A^*\mathcal{G}$ .

When  $\mathcal{G} \rightrightarrows P$  is a Poisson Lie group  $G$  (with  $P = \{\cdot\}$ ),  $T^*AG$  reduces as a manifold to  $\mathfrak{g} \times \mathfrak{g}^*$ . The vertical Lie algebroid structure arises by lifting the Lie algebra structure on  $\mathfrak{g}$  to the pullback vector bundle  $\mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathfrak{g}^*$ ; similarly with  $\mathfrak{g}^*$ . In turn the classical Drinfel’d double structure on  $\mathfrak{g} \oplus \mathfrak{g}^*$  may be obtained from  $T^*A\mathcal{G}$  by a process of diagonalization [19]. Thus Figure 3(b) may be regarded as an extension to Poisson groupoids of the classical Drinfel’d double of a Poisson Lie group.

The relationship between the Lie algebroid structures in diagrams such as those in Figure 3(a)(b) cannot be characterized in the same way as for  $\mathcal{VB}$ -groupoids and  $\mathcal{LA}$ -groupoids; it is not possible to say that the Lie algebroid brackets on the (say) vertical structures are morphisms with respect to the horizontal structures. This problem can be solved by using the duality theory

of double vector bundles. Very briefly, given a double vector bundle as in Figure 3(c) with Lie algebroid structures on each side, assume that each Lie algebroid structure on  $D$  is linear with respect to the other; then dualizing each structure on  $D$  leads to a pair of Poisson vector bundles in duality, and  $D$  is defined to be a *double Lie algebroid* if the corresponding dual Lie algebroids form a Lie bialgebroid. A full account is in [19]. The case in which one pair of parallel Lie algebroids is zero is variously called an  $\mathcal{LA}$ -vector-bundle or a  $\mathcal{VB}$ -algebroid. Thus one may regard a double Lie algebroid as a double vector bundle admitting a horizontal  $\mathcal{VB}$ -algebroid structure and a vertical  $\mathcal{VB}$ -algebroid structure which are suitably compatible [10].

Any vector bundle  $A \rightarrow M$  gives rise to a *cotangent double vector bundle*  $T^*A$  as shown in Figure 3(d). The structure  $T^*A \rightarrow A^*$  arises from using the canonical diffeomorphism  $T^*A \cong T^*A^*$  [20]. Now suppose that both  $A$  and  $A^*$  have Lie algebroid structures, not necessarily related. The Lie algebroid structure on  $A^*$  induces a Poisson structure on  $A$  and this induces a Lie algebroid structure on  $T^*A \rightarrow A$ . Likewise the Lie algebroid structure on  $A$  induces a Lie algebroid structure on  $T^*A^* \rightarrow A^*$ . These structures make  $T^*A \cong T^*A^*$  a double Lie algebroid if and only if  $A$  and  $A^*$  form a Lie bialgebroid [19].

Thus it is reasonable to argue that the cotangent double Figure 3(d) plays the role for Lie bialgebroids which is played for Lie bialgebras by the classical Drinfel'd double. In particular the bialgebroid equation (34) is encapsulated in the double Lie algebroid conditions for the cotangent double  $T^*A$ .

The notion of double Lie algebroid provides an alternative to the extension of the classical Drinfel'd double to Lie bialgebroids of Liu, Weinstein and Xu [14]. The theory of Courant algebroids has led to the important concept of Dirac structures, whereas double Lie algebroids arise as second-order invariants of double Lie groupoids. We will describe this briefly below.

Double Lie algebroids have been defined in terms of supergeometry by Th. Voronov [27], and this is an exceptionally clear and elegant formulation. In the late 1990s, Vaintrob showed that a homological vector field of degree  $+1$  on a supermanifold is equivalent to a Lie algebroid structure on the parity reversed bundle. A double Lie algebroid is then defined by two homological vector fields, of suitable weights, on a double parity reversion, provided that the homological vector fields *commute*. This includes the construction of the super cotangent double by Roytenberg [25].

Very recently a formulation of double Lie algebroids in terms of representations up to homotopy has been given by Gracia-Saz, Jotz Lean, Mehta and the author [9].

It was shown in [21] that given a Lie bialgebroid  $(A, A^*)$  in which  $A = A\mathcal{G}$  is the Lie algebroid of a source-simply-connected Lie groupoid  $\mathcal{G}$ , there is a Poisson structure on  $\mathcal{G}$  making it a Poisson groupoid, and inducing the given Lie algebroid structure on  $A^*$ . In terms of double structures, this shows that the cotangent double Lie algebroid  $T^*A\mathcal{G}$  may be integrated to an  $\mathcal{LA}$ -groupoid  $T^*\mathcal{G}$ . Very recently Bursztyn, Cabrera and del Hoyo [1] have given a general integrability result for double Lie algebroids in terms of  $\mathcal{LA}$ -groupoids.

We turn now to the question of integrating an  $\mathcal{LA}$ -groupoid, such as that in Figure 2(a). It was shown by Lu and Weinstein [15] that underlying any Poisson Lie group is a symplectic double groupoid.

In general a *double Lie groupoid* consists of a manifold  $S$  with two Lie groupoid structures,

$S \rightrightarrows H$  and  $S \rightrightarrows V$ , where  $H$  and  $V$  are themselves Lie groupoids over a manifold  $M$ , as shown in Figure 4(a), such that the structure maps of each groupoid structure are morphisms with respect to the other structure, and satisfying a double source condition analogous to those in Definitions 5.1 and 5.2.

In the same way that elements of a groupoid are visualized as arrows, elements of a double groupoid are visualized as squares, the horizontal edges of which come from  $H$ , the vertical edges from  $V$ , and the four corners from  $M$ ; see Figure 4(b)(c). If  $v'_1 = v_2$  one can compose the two elements shown horizontally — the vertical edges of the composite, shown in Figure 4(d), are determined by the groupoid axioms for  $S \rightrightarrows V$  and the horizontal edges by the condition that the source and target for  $S \rightrightarrows H$  are morphisms.

The condition that each groupoid multiplication is a morphism with respect to the other structure is an interchange law: given four squares which can be arranged to form a larger square with matching inner edges, the result of composing vertically and then horizontally is the same as composing horizontally and then vertically.

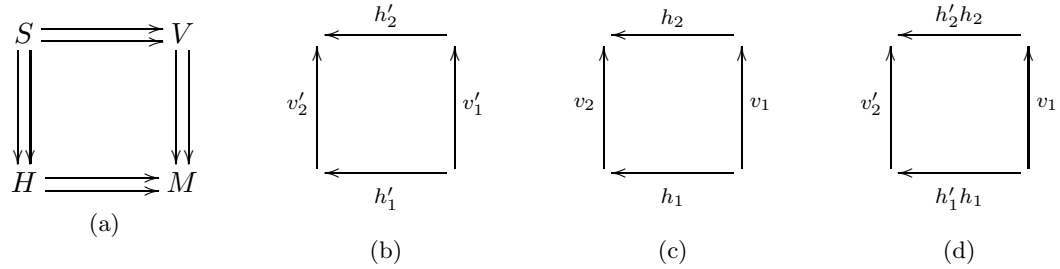


Figure 4.

Given a double Lie groupoid one may apply the Lie functor to the horizontal structures and obtain an  $\mathcal{LA}$ -groupoid as in Figure 5(a). As described earlier, the Lie functor may then be applied to the vertical structures to give the double Lie algebroid  $A_V A_H S$  of  $S$ ; see Figure 5(b). Equally, one may apply the Lie functor to the vertical structures and obtain an  $\mathcal{LA}$ -groupoid as in Figure 5(c), and then to the horizontal structures to give the double Lie algebroid  $A_H A_V S$  of  $S$ ; see Figure 5(d). There is a canonical diffeomorphism from  $A_V A_H S$  to  $A_H A_V S$ .

For example if  $S = M^4$ , one obtains the  $\mathcal{LA}$ -groupoid  $TM \times TM$  and the double Lie algebroid is the iterated tangent bundle  $T^2 M$ .

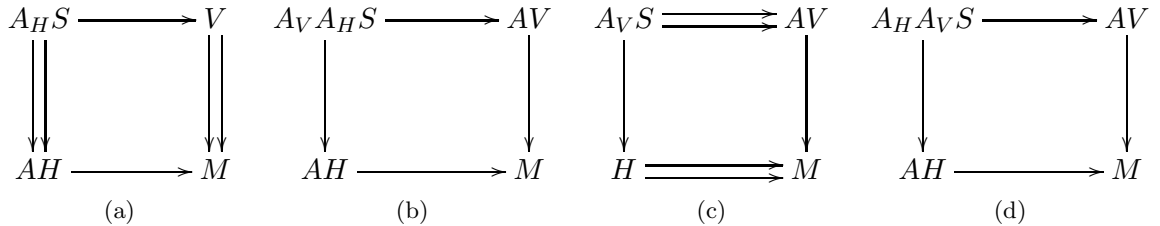


Figure 5.

Now consider a Poisson Lie group  $G$  and let  $D$  denote the simply-connected Lie group cor-

responding to the classical Drinfel'd double  $\mathfrak{g} \bowtie \mathfrak{g}^*$ . Then the inclusions  $\mathfrak{g} \rightarrow \mathfrak{g} \bowtie \mathfrak{g}^*$  and  $\mathfrak{g}^* \rightarrow \mathfrak{g} \bowtie \mathfrak{g}^*$  induce morphisms  $G \rightarrow D$ ,  $g \mapsto \bar{g}$  and  $G^* \rightarrow D$ ,  $\varphi \mapsto \bar{\varphi}$ , where  $G^*$  is the simply-connected Lie group corresponding to  $\mathfrak{g}^*$ . Lu and Weinstein [15] define a double Lie groupoid  $S$  as shown in Figure 6(a), for which the elements are quadruples  $(g_2, g_1, \varphi_2, \varphi_1) \in G \times G \times G^* \times G^*$  such that  $\bar{g}_2 \bar{\varphi}_1 = \bar{\varphi}_2 \bar{g}_1 \in D$ , as shown in Figure 6(b).

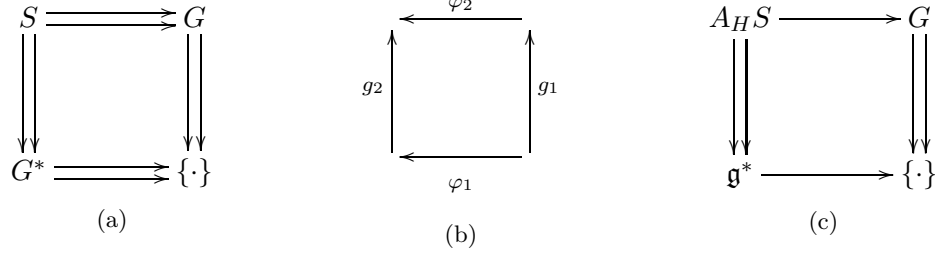


Figure 6.

The Lie algebra structure on  $\mathfrak{g} \bowtie \mathfrak{g}^*$  induces a symplectic structure on  $S$  which makes  $S$  a symplectic groupoid with respect to both groupoid structures. It then follows that  $A_H S \cong T^*G$  as Lie algebroids and as Lie groupoids. In effect, [15] has integrated the  $\mathcal{LA}$ -groupoid in Figure 2(a), in the case of Poisson Lie groups, to the symplectic double groupoid  $S$ .

A general integration result, starting with an  $\mathcal{LA}$ -groupoid  $\Omega$  as in Figure 2(b), where  $\Omega \rightarrow \mathcal{G}$  is the Lie algebroid of a suitable Lie groupoid  $\Gamma \rightrightarrows \mathcal{G}$ , and which constructs a second Lie groupoid structure  $\Gamma \rightrightarrows \mathcal{H}$ , with  $A\mathcal{H} \cong A$ , so that  $\Gamma$  becomes a double Lie groupoid over  $\mathcal{G}$  and  $\mathcal{H}$ , seems very far from being accessible.

## Endword

In addition to works already cited, other valuable sources on Poisson geometry include the books by Vaisman [26], Cannas da Silva and Weinstein [2], Dufour and Zung [8], and Laurent-Gengoux et al [13].

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